

# TOWARDS THE KPP-PROBLEM AND $\log t$ -FRONT SHIFT FOR HIGHER-ORDER NONLINEAR PDES II. QUASILINEAR BI- AND TRI-HARMONIC EQUATIONS

V.A. GALAKTIONOV

ABSTRACT. Extensions of ideas of Kolmogorov, Petrovskii, and Piskunov (1937) [38] on travelling wave propagation in the reaction-diffusion equation

$$u_t = u_{xx} + u(1 - u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad u_0(x) = H(-x) \equiv \{1 \text{ for } x < 0; 0 \text{ for } x \geq 0\},$$

$H(\cdot)$  being the Heaviside function, are discussed. The present paper continues the study began in [23] for higher-order semilinear *bi-harmonic* and *tri-harmonic* equations

$$u_t = -u_{xxxx} + u(1 - u) \quad \text{and} \quad u_t = u_{xxxxxx} + u(1 - u), \quad \text{etc.}$$

Here, some of the results are extended to the corresponding *quasilinear* degenerate parabolic models with nonlinearities of the *porous medium type* ( $n > 0$ ),

$$u_t = -(|u|^n u)_{xxxx} + u(1 - u), \quad u_t = (|u|^n u)_{xxxxxx} + u(1 - u), \quad \text{etc.}$$

Two main questions to discuss are:

- (i) existence of travelling waves via any analytical/numerical methods, and
- (ii) their stability and derivation of the  $\log t$ -shifting of moving fronts for some class of data  $u_0$  (not for  $H(-x)$ ).

## 1. INTRODUCTION: THE CLASSIC KPP-PROBLEM, ITS QUASILINEAR EXTENSIONS, AND HIGHER-ORDER QUASILINEAR PARABOLIC PDES

We begin with an introduction concerning classic results; see more details in [23, § 1].

**1.1. The classic KPP-problem of 1937: convergence to TWs.** In the KPP-problem [38] (1937)

$$(1.1) \quad u_t = u_{xx} + u(1 - u), \quad x \in \mathbb{R}, \quad t > 0; \quad u(x, 0) = u_0(x) \text{ in } \mathbb{R},$$

with the step (Heaviside) initial function

$$(1.2) \quad u_0(x) = H(-x) \equiv \begin{cases} 1, & x < 0; \\ 0, & x \geq 0, \end{cases}$$

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the solution was proved to converge to the so-called *minimal* travelling wave (TW) solution corresponding to the minimally possible TW speed

$$(1.3) \quad \text{the minimal speed of TW propagation : } \lambda_0 = 2.$$

Looking for a TW profile  $f(y)$  for arbitrary  $\lambda > 0$  yields:

$$(1.4) \quad \begin{cases} u_*(x, t) = f(y), \ y = x - \lambda t, \ \text{where } f \text{ solves the ODE problem} \\ -\lambda f' = f'' + f(1 - f), \ y \in \mathbb{R}; \quad f(-\infty) = 1, \ f(+\infty) = 0. \end{cases}$$

This 2nd-order ODE, on the phase-plane  $\{f, f'\}$ , by setting  $f' = P(f)$ , reduces its order:

$$(1.5) \quad \frac{dP}{df} = -\lambda - \frac{f(1-f)}{P},$$

and it was shown that there exists the *minimal* speed  $\lambda_0 = 2$  and the unique (up to translation) *minimal* TW profile  $f(y)$ . Using the natural normalization

$$(1.6) \quad f(0) = \frac{1}{2}$$

this minimal TW profile is defined uniquely. In addition,

$$(1.7) \quad f'(y) < 0 \quad \text{in } \mathbb{R}.$$

The characteristic equation for the linearized operator in (1.4) has a multiple zero:

$$(1.8) \quad g'' + 2g' + g = 0 \quad \text{and} \quad g = e^{\mu y} \implies (\mu + 1)^2 = 0 \implies \mu_{1,2} = -1,$$

that yields the following asymptotic behaviour of  $f(y)$ :

$$(1.9) \quad f(y) = C_0 y e^{-y}(1 + o(1)) \quad \text{as } y \rightarrow +\infty, \quad \text{where } C_0 > 0 \text{ is a constant.}$$

Concerning the relation between the ODE TW problem (1.5) for  $\lambda_0 = 2$  and the PDE Cauchy one (1.1), (1.2), the novel remarkable analysis in [38] of convergence as  $t \rightarrow +\infty$  of the solution of the Cauchy problem (1.1), (1.2) to the minimal TW (1.4) was performed in the TW moving frame. This was very essential, and not in view of the obvious  $x$ -translational invariance of the equation (1.1); see below. Eventually, using PDE methods, the KPP-authors proved that the TW front moves like

$$(1.10) \quad x_f(t) = 2t - g(t) \quad \text{as } t \rightarrow \infty, \quad \text{with } g'(t) \rightarrow 0,$$

where the front location  $x_f(t)$  is uniquely determined from the equation

$$(1.11) \quad u(x_f(t), t) = \frac{1}{2} \quad \text{for all } t \geq 0.$$

Then the convergence result of [38] takes the form:

$$(1.12) \quad u(x_f(t) + y, t) \rightarrow f(y) \quad \text{as } t \rightarrow +\infty \quad \text{uniformly in } y \in \mathbb{R}.$$

**1.2. Bramson’s log  $t$ -front drift.** The next important question is the actual behaviour of the TW shift  $g(t)$  in (1.10) for  $t \gg 1$  (not addressed in [38]). This is about an “centre subspace (manifold) drift” of general solutions of the KPP PDE (1.1) along a one-parameter family of exact ODE TWs  $\{f(y + a), a \in \mathbb{R}\}$ .

This open problem was solved in 1983 by Bramson [10] (see also [29]) by using pure probabilistic techniques (the Feynman–Kac integral formula together with sample path estimates for Brownian motion). It was proved that, within the PDE setting (1.1), (1.2), there is an *unbounded* log  $t$ -shift of the moving TW front:

$$(1.13) \quad g(t) = k \log t (1 + o(1)) \quad \text{as } t \rightarrow +\infty, \quad \text{with } k = \frac{3}{2},$$

Thus, (1.13) implies eventual, as  $t \rightarrow +\infty$ , *infinite* retarding of the solution  $u(x, t)$  from the corresponding minimal TW (uniquely fixed by (1.6)), though the convergence (1.12) takes place in the TW frame.

**1.3. Known extensions: TW profiles for quasilinear second-order reaction-diffusion equations.** The first attempts to extend the KPP results to quasilinear second-order reaction-diffusion equations

$$(1.14) \quad u_t = (k(u)u_x)_x + Q(u), \quad Q(0) = Q(1) = 0, \quad k(u) \geq 0,$$

including the porous medium diffusion operators with  $k(u) = u^n$ ,  $n > 0$ , were known since the 1970s<sup>1</sup>; see results and extra references in [1, 3, 2, 17, 18, 30, 31, 32, 33, 34, 37, 41, 42], [44, p. 34], [45, 46, 48], etc.

However, it seems that any results concerning possible log  $t$ -front shift for quasilinear KPP problems including those with the degenerate operators of the PME type

$$(1.15) \quad u_t = (u^{n+1})_{xx} + Q(u), \quad Q(0) = Q(1) = 0, \quad n > 0 \quad (u \geq 0)$$

(with, say, positive non-Heaviside data; then  $+\log t$ -shift may occur) were still unknown. Recall that, since (1.15) describes finite propagation, for the Heaviside initial data, the minimal travelling wave is indeed stable (just by a standard barrier-comparison approach, this was known since 1980s; see some references above). In this stability sense, the quasilinear KPP-2 problem (1.15) is simpler than the classic one for  $n = 0$ . However, we claim that log  $t$ -shift is possible for (1.15) for some classes of initial data. Indeed, by obvious reasons, in order to avoid a trivial comparison from below with a sufficiently shifted minimal TW  $f(x - \lambda_0 t + a)$ , with  $a \gg 1$ , one needs that  $u_0(x)$  must intersect ALL such TW profiles  $f(x + a)$  (evidently, the Heaviside one does not do that). Actually, this means that, for (1.15), a “correct” KPP-type setting (a really difficult one) assumes that initial data *do not* have finite interface and are specially distributed, that do not allow to use straightforward upper/lower barriers for TW-like estimates above and below.

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<sup>1</sup>It seems, the first ever sufficiently general existence TW result for (1.14) was obtained by A. Nepomnyashchy, which was published in Russian in the technical Journal “Voprosy Atomnoi Nauki i Tehniki” (“Proceedings of Atomic Science and Technology”) and was not translated into English.

If fact, it is easy to construct a number of explicit TW solutions for the PME with special source terms in (1.15) using the *pressure variable*:

$$(1.16) \quad u^n = v \implies v_t = (n+1)vv_{xx} + \frac{n+1}{n}(v_x)^2 + q(v), \quad q(v) = nv^{\frac{n-1}{n}}Q(v^{\frac{1}{n}}).$$

The ODE for TW profiles  $f(y)$  then takes the form:

$$(1.17) \quad -\lambda f' = (n+1)ff'' + \frac{n+1}{n}(f')^2 + q(f),$$

and, as usual, we are looking for a solution  $f$  having finite interface for  $y > 0$  and  $f \rightarrow 1$  as  $y \rightarrow -\infty$ , with a linear spatial behaviour at the interfaces (this well corresponds to Darcy's law of their propagation for such weak solutions).

**Example 1.** There exists the following explicit solution of (1.17) ( $(\cdot)_+$  denotes the positive part):

$$(1.18) \quad \begin{aligned} f(y) &= \frac{(-y)_+}{1-y}, \quad \lambda_0 = \frac{n+1}{n}, \quad \text{where} \\ q(f) &= \lambda(f-1)^2 - 2(n+1)(f-1)^3 - (2(n+1) + \frac{n+1}{n})(f-1)^4. \end{aligned}$$

**Example 2.** There exists the following explicit solution of (1.17):

$$(1.19) \quad \begin{aligned} f(y) &= \frac{(e^{-y}-1)_+}{e^{-y}+1}, \quad \lambda_0 = \frac{n+1}{2n}, \quad \text{where} \quad q(f) = \frac{1}{2}(1-f)[\lambda(1+f) \\ &\quad -(n+1)f(1-f) + (n+1)f(1+f)^2 - \frac{1}{2}\frac{n+1}{n}(1-f)(1+f)^2]. \end{aligned}$$

We claim that, in the corresponding PDE framework, such explicit TWs admit a  $\pm \log t$ -front shifting along an “affine centre subspace”, for suitable classes of initial data ( $+\log t$  for positive data, and  $-\log t$  – for changing sign one). The derivations is similar (and simpler) to that in Section 9 for the quasilinear bi- and tri-harmonic flows.

**1.4. The present goal: extensions of the KPP–problem to higher-order quasilinear parabolic PDEs.** The main goal of the present research comprising [23, 24], and the present paper, is to show that the KPP–ideology can be extended to a variety of other more complicated higher-order semilinear and quasilinear PDEs with source-type terms.

Thus, our goal here is to continue such a study, and, using various analytic, formal, and numerical methods), to show that such a general viewing of the KPP ideas make sense, and that many higher-order PDEs inherits some (but never all, of course) key features of this classic analysis.

Let us note that, in [8, 9] (see also a large list of references therein, as well as those in the present paper), TW profiles were studied to a class of quasilinear *thin film equations*<sup>2</sup>

$$(1.20) \quad u_t + (f(u))_x = (b(u)u_x)_x - (c(u)u_{xxx})_x.$$

In particular, for more general TFE-type equations in  $\mathbb{R}^N \times \mathbb{R}_+$ , stability of 1D TWs was studied in [35], though, as in most of the papers mentioned around, the authors dealt with nonnegative solutions of a free-boundary value problem, rather than the Cauchy one.

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<sup>2</sup>Indeed, this quasilinear model is most relevant to the present study of (1.21) and (1.29), though our solutions of the Cauchy problems *are not* nonnegative, are oscillatory, and of changing sign near the right-hand finite interface.

Thus, firstly, we will discuss some aspects of KPP-type problems for higher-order quasilinear partial differential equations (PDEs), with the same Heaviside initial data. From applications, such KPP-type problems deal with nonlinear higher-order diffusion operators leading to well-known nowadays models; see references on various fourth and  $2m$ th-order semilinear and quasilinear PDEs in [8, 9, 11, 14, 16, 19, 20].

To this end, in the present paper, we extend some of the KPP-type results to the *quasilinear* bi-harmonic equation with the diffusion operators of the fourth-order *porous medium* type

$$(1.21) \quad u_t = -(|u|^n u)_{xxxx} + u(1 - u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad \text{where } n > 0.$$

The corresponding TW with the speed of propagation  $\lambda$  is then governed by the following fourth-order ODE:

$$(1.22) \quad u_*(x, t) = f(y), \quad y = x - \lambda t \implies -\lambda f' = -(|f|^n f) f^{(4)} + f(1 - f),$$

with the singular boundary conditions at infinity:

$$(1.23) \quad f(y) = 0 \text{ for } y \gg 1 \quad \text{and} \quad f(y) \rightarrow 1 \text{ as } y \rightarrow -\infty \text{ exponentially fast.}$$

The first condition in (1.23) just takes into account that the *degenerate* equation (1.21) describes processes with *finite propagation* of disturbances, a very well known fact for such PDEs; see key references and results in [22]. Moreover, solutions are oscillatory and changing sign close to finite interfaces. Therefore, the following example of a nonnegative TW solution cannot be generic:

**Example 3: nonnegative smooth TW for a bi-harmonic flow.** We take  $n = 1$  and consider the fourth-order parabolic equation with a TW  $f \geq 0$ :

$$(1.24) \quad u_t = -(u^2)_{xxxx} + q(u) \implies -\lambda f' = -(f^2)^{(4)} + q(f).$$

For a special choice of the reaction  $q$  (see below), it admits the following explicit solution:

$$(1.25) \quad f(y) = \left[ \frac{(-y)}{1-y} \right]^3,$$

where the non-oscillatory behaviour  $f(y) \sim (-y)^3$  close to the interface  $y = 0^-$  is the actual decay for weak solutions of the Cauchy problem for  $n = 1$ ; see Section 4. It follows from (1.25) that

$$(1.26) \quad (-y) = \frac{1}{1-f^{1/3}} - 1,$$

so that, substituting (1.25) into the ODE in (1.24), performing all the differentiations and eventually expressing  $(-y)$  in terms of  $f$  via (1.26), one obtains the actual source term  $q(f)$ , for which this is a solution. The required value of the speed  $\lambda$  is then obtained from the asymptotic analysis near the interface, as  $y \rightarrow 0^-$ , as follows:

$$(1.27) \quad f(y) = (-y)^3(1 + O(y)) \implies 3\lambda y^2 = -(y^6)^{(4)} + q(f) + \dots, \text{ so } \lambda_0 = -120 < 0.$$

Indeed, for other values of  $\lambda$ , when the terms  $O(y^2)$  are not cancelled in the equation in (1.27), one obtains a sufficiently “singular” reaction term  $q(f)$ :

$$(1.28) \quad q(f) \sim (-y)^2 \sim f^{\frac{2}{3}} \quad \text{as } y \rightarrow 0^-,$$

i.e.,  $q(f)$  is *not* Lipschitz continuous at  $f = 0$ , so that, in the PDE setting, there occurs a typical problem of non-uniqueness of solutions (if  $q(f)$  is a source term for  $f \approx 0$ ).

However, we will show in Section 4 that the nonnegative TWs such as (1.25) are not generic in the sense that a.a. solutions are oscillatory and changing sign near finite interfaces. In other words, nonnegativity of solutions is not an invariant property of such quasilinear bi-harmonic flows in the Cauchy problem setting (but can be in a FBP one, what was first shown by Bernis–Friedman [6] for quasilinear thin film equations). Therefore, we must neglect any opportunity to deal with nonnegative solutions of such equations with arbitrary sufficiently smooth coefficients.

For any  $\lambda \neq 0$ , the problem (1.22), (1.23) is of the elliptic type, but *it is not variational*. Therefore, as in [23, 24], one cannot use advanced methods for higher-order ODEs with potential operators associated with homotopy-hodograph and other approaches [43, 36, 47] and/or Lusternik–Schnirel’man and fibering theory [26, 27]. Therefore, the ODE (1.22), though looking rather simple, and, at least, simpler than most of related fourth-order ODEs already studied in detail, represents a serious challenge and cannot be tackled directly by known tools of modern nonlinear analysis and operator theory.

Finally, we study TWs for the *quasilinear tri-Harmonic equation*

$$(1.29) \quad u_t = (|u|^n u)_{xxxxx} + u(1 - u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad \text{where } n > 0.$$

**1.5. Other related results of the present research.** The previous paper [23] is devoted to the *semilinear parabolic equations* for  $n = 0$ , i.e.,

$$(1.30) \quad u_t = -u_{xxxx} + u(1 - u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \implies -\lambda f' = -f^{(4)} + f(1 - f),$$

and the boundary conditions now read

$$(1.31) \quad f(y) \rightarrow 0 \quad \text{and} \quad f(y) \rightarrow 1 \quad \text{as } y \rightarrow -\infty \quad \text{“maximally” exponentially fast.}$$

This “maximal” decay of  $f(y)$  at infinity somehow includes some kind of the remnants of a “minimality” of the possible TW profiles, though any direct specification of such a property is difficult to express rigorously.

Thus, the present paper and [23] deal with higher-order *parabolic* equations of reaction-diffusion type, while, in the third part of this research [24], we study KPP-type problems for other types of PDEs including *dispersion*, *hyperbolic*, and other ones. Namely, in [24], we will deal with higher-order hyperbolic and dispersion equations such as

$$(1.32) \quad u_{tt} = -u_{xxxx} + u(1 - u) \quad \text{and} \quad u_{ttt} = D_x^{(10)} u + u(1 - u), \quad \text{etc.}$$

As for more “exotic” PDE models, as a formal but quite illustrative example, we consider, in [24], higher-order dispersion equations and end up with the following one:

$$(1.33) \quad D_t^9 u = D_x^{11} u + u(1 - u),$$

with eleventh-order ODE for the TW profiles

$$(1.34) \quad -\lambda^9 f^{(9)} = f^{(11)} + f(1 - f) \quad \text{in } \mathbb{R} \quad (\text{plus (1.23)}).$$

We also consider in [24] an example of a *quasilinear* dispersion equation with a similar nonlinearity.

**1.6. Main questions to study.** Thus, for quasilinear KPP-problems (1.21) and (1.29), the main questions to study, here and in [23, 24] (though in the quasilinear cases the results achieved are weaker), are:

**(I) THE PROBLEM OF TW EXISTENCE:** existence of travelling waves via analytical/numerical methods. Firstly, we easily show the positivity:  $\lambda > 0$  always.

**(II) THE  $\log t$ -SHIFT PROBLEM:** given some speed  $\lambda_0 > 0$ , derivation of a possible  $\log t$ -shifting of the moving front in the problem (1.1), for classes of initial data  $u_0(x)$  (for Heaviside data (1.2), we expect no shift and convergence to a TW; cf. some convincing stability results in [8, 9, 35] for thin film equations). As in the semilinear cases, this phenomenon is also connected with a kind of an “(affine) centre subspace behaviour” for the rescaled equation.

We thus show that  $\log t$ -shifting phenomena, for some classes of data) are also available for quasilinear degenerate KPP-4 and other problems, though is less generic than for semilinear higher-order equations, since require, in general, *still unknown* hypotheses on  $u_0(x)$ .

Unlike the semilinear analytic PDEs for  $n = 0$  in [23], the rescaled quasilinear equations for  $n > 0$  are not analytic, so we cannot proceed with a deeper study of the omega limits for such equations.

## 2. THE BASIC HIGHER-ORDER KPP-4N PROBLEM

Consider the basic KPP-(4,1) $n$  (or simply KPP-4 $n$ , that cannot confuse in the parabolic case) problem (1.30), and let us begin with its ODE counterpart (1.22), (1.23).

**2.1.  $\lambda$  is always positive.** As in [23, § 2], we first prove a first simple result on the positivity of admissible speeds  $\lambda$ :

### Proposition 2.1.

(2.1) *If there exists a solution  $f(y) \not\equiv 0$  of (1.22), (1.23), then  $\lambda > 0$ .*

*Proof.* Multiplying the ODE (1.22) by  $(|f|^n f)'$  and integrating by parts over  $\mathbb{R}$  and using the boundary conditions (1.23) yield

$$(2.2) \quad -\lambda(n+1) \int |f|^n (f')^2 = [F(f(y))]_{-\infty}^{+\infty} = -F(1) < 0,$$

where  $F(f) = (n+1) \int_0^f |z|^n z(1-z) dz$ .  $\square$

**2.2. Numerical construction of TW profiles.** Again, as in [23], we begin with presenting first numerical results, which directly show the global structure of such TW profiles to be, at least partially, justified analytically. We use the **bvp4c** solver of the **MatLab** with a sufficient accuracy; see more details in [23, § 2]. Note that, as the initial data for further iterations, we always took the Heaviside function

(2.3)  $\text{initial data for numerical iterations are often } H(-y),$

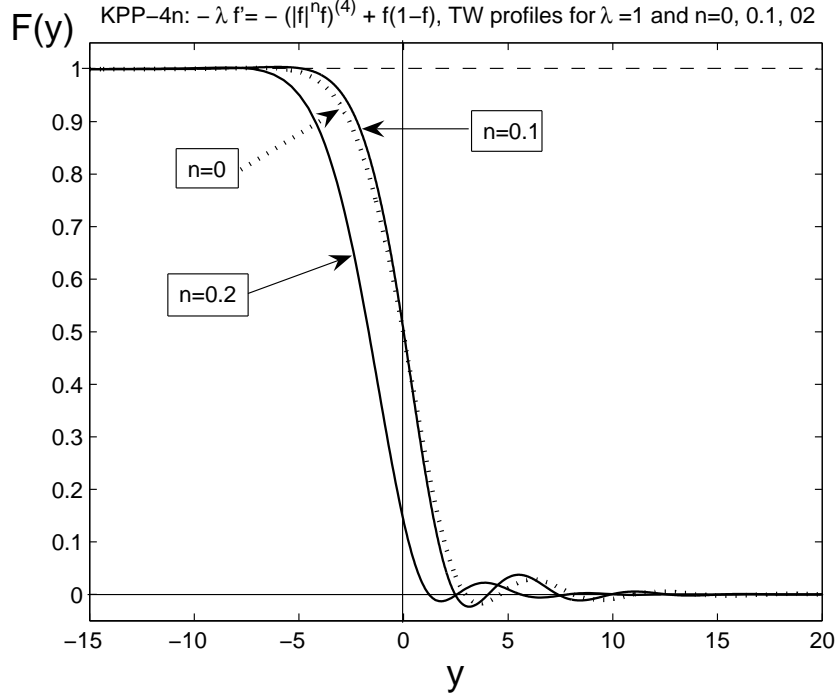


FIGURE 1. The TW profiles  $F(y)$  of (2.4), (1.23) for  $\lambda = 1$ ,  $n = 0, 0.1, 0.2$ .

i.e., as in (1.2). This once more had to help us to converge to a proper “minimal” profile (indeed, there are many other TW profiles), though, of course, this was not guaranteed *a priori*. We kept this rule for all other KPP- $(k, l)$  problems of interest in [23, 24].

For convenience, we perform all the calculations for the function

$$(2.4) \quad F(y) = |f(y)|^n f(y) \implies F^{(4)} = \frac{\lambda}{n+1} |F|^{-\frac{n}{n+1}} F' + |F|^{-\frac{n}{n+1}} F (1 - |F|^{-\frac{n}{n+1}} F),$$

so that the ODE in (1.22) becomes *semilinear*.

Figure 1 shows TW profiles  $F$  for  $\lambda = 1$  for sufficiently small  $n = 0, 0.1, 0.2$ . The semilinear case  $n = 0$  was used to get a good comparison with the results in [23, § 2]. For larger  $n = 0.8$ , with the same  $\lambda = 1$ , the numerical results are shown in Figure 2.

We next fix  $n = 1$  and  $\lambda = 0.5$  as Figure 3 shows. A more detailed structure of oscillations of solutions about equilibria  $F = 0$  (a) and  $F = 1$  (b) is presented in Figure 4. Separately, Figure 5 shows TW profiles  $F(y)$  as in (2.4) for  $n = 1$  and  $\lambda = \frac{1}{4}$ .

Finally, we must admit that, for  $n > 0$ , and, especially, for larger values  $n \sim 1$ , it is much more difficult to get reliable well-converging numerical results unlike the more straightforward semilinear case  $n = 0$ , [23]. Many our attempts, even rather time-consuming (lasting for several hours of the **MatLab**) led to oscillatory non-converging profiles and/or to singular Jacobians (a kind of “blow-up” of computations).

However, our numerical experiments produced a rather convincing evidence of existence of TW profiles for a variety of  $n > 0$  and velocities  $\lambda > 0$ , which we will need to develop further.



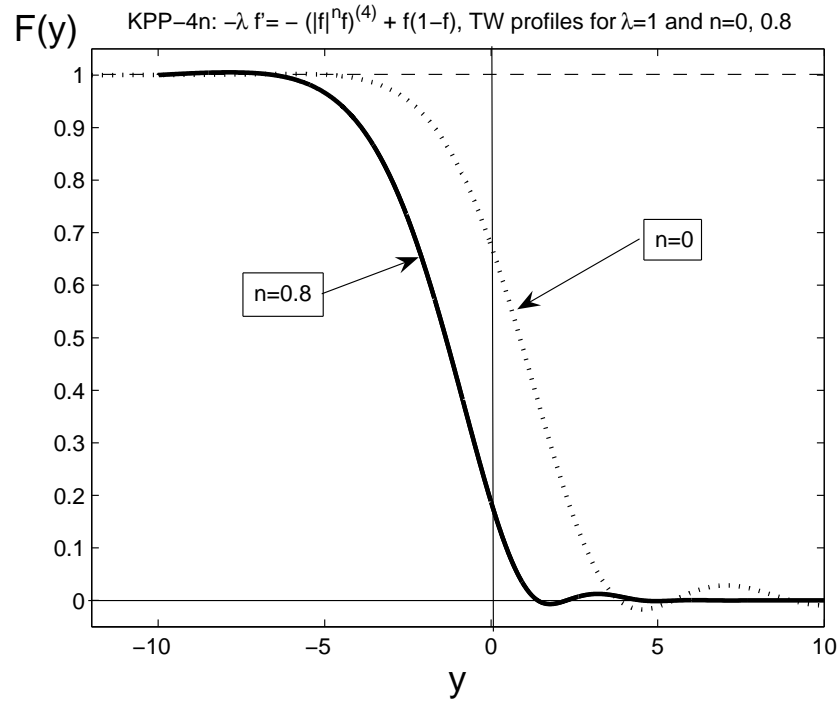


FIGURE 2. The TW profiles  $F(y)$  of (2.4), (1.23) for  $\lambda = 1$ ,  $n = 0.8$ .

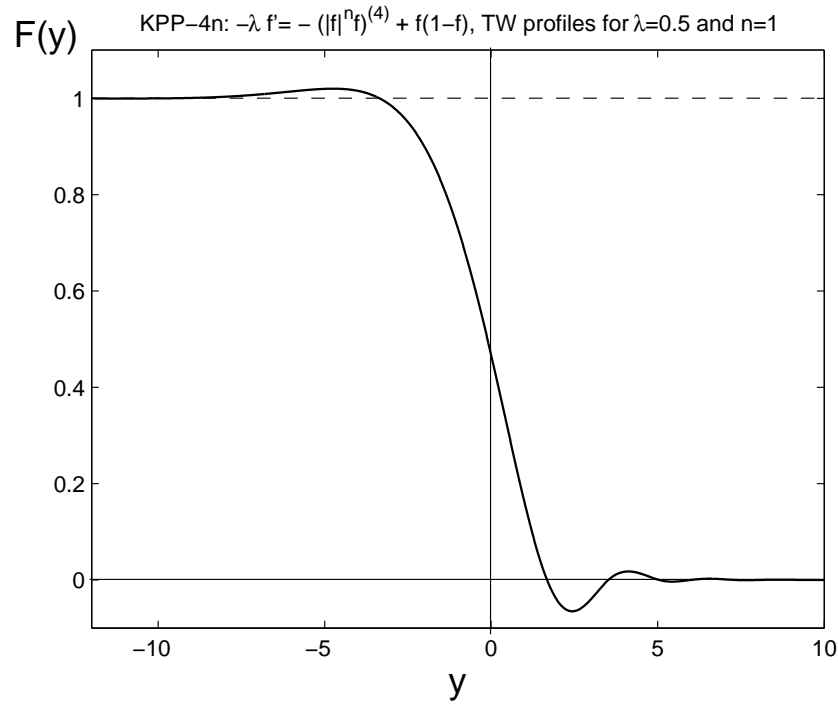
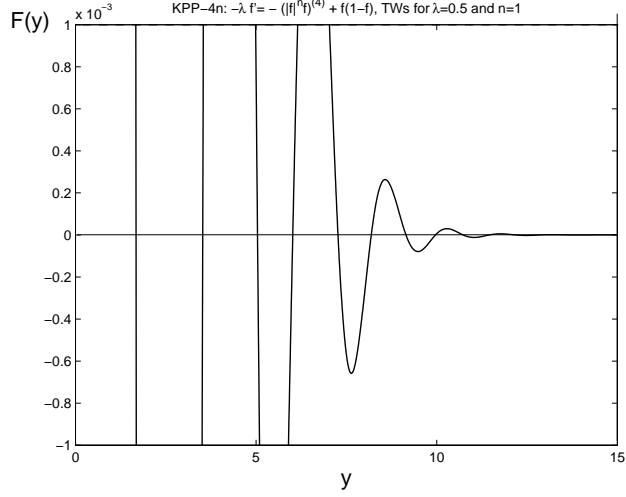
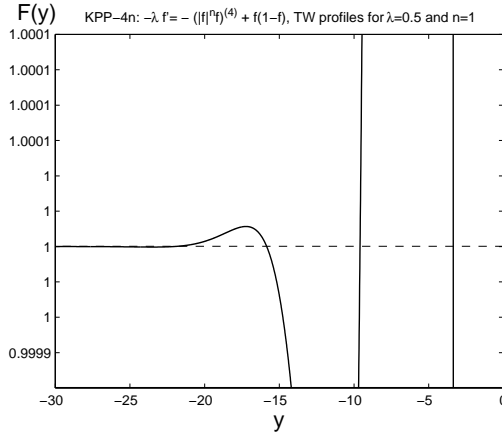


FIGURE 3. The TW profiles  $F(y)$  of (2.4), (1.23) for  $\lambda = 0.5$ ,  $n = 1$ .



(a) oscillations about  $F = 0$



(b) oscillations about  $F = 1$

FIGURE 4. An oscillatory convergence of the TW profile  $F(y)$  to 0,  $y \gg 1$  (a) and to 1,  $y \ll -1$ ;  $\lambda = 0.5$  and  $n = 1$ .

Thus, we begin with necessary local analysis of the behaviour of TW profiles  $f(y)$  close to equilibria  $f = 1$  and  $f = 0$  (see the next section), in order to fix dimensions of their stable and unstable manifolds, for further matching.

### 3. THE 2D STABLE BUNDLE AS $y \rightarrow -\infty$ AND INSTABILITIES

This analysis is not much different from that for  $n = 0$  in [23, § 2.3].

**3.1. Linearization about  $f = 1$ .** Thus, setting  $f = 1 + g$  in (1.22) and linearizing yield the following characteristic equation:

$$(3.1) \quad -\lambda g' = (n+1)g^{(4)} + g \quad \text{and} \quad g(y) = e^{\mu y} \implies H_-(\mu, \lambda) \equiv (n+1)\mu^4 - \lambda\mu + 1 = 0.$$

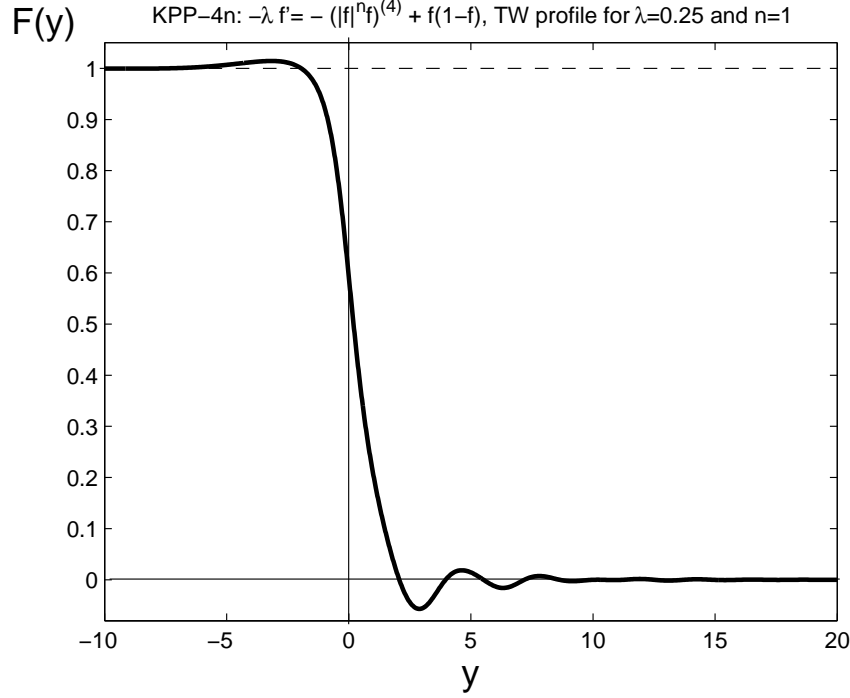


FIGURE 5. The TW profiles  $F(y)$  of (2.4), (1.23) for  $\lambda = 0.25$ ,  $n = 1$ .

Therefore, for  $\lambda = 0$ , we have 2D stable ( $\text{Re}(\cdot) > 0$ ) and unstable ( $\text{Re}(\cdot) < 0$ ) bundles with the roots

$$(3.2) \quad \mu_{\pm}(0) = \pm \frac{1+i}{\sqrt{2}}(n+1)^{-\frac{1}{4}} \quad \text{and} \quad \bar{\mu}_{\pm}(0) = \pm \frac{1-i}{\sqrt{2}}(n+1)^{-\frac{1}{4}}.$$

By continuity, for all small  $\lambda > 0$ , there exists 2D stable manifold of the equilibrium 1 with the roots

$$(3.3) \quad \text{stable bundle : } \mu_{+}(\lambda) \quad \text{and} \quad \bar{\mu}_{+}(\lambda).$$

**Proposition 3.1.** *For any  $n \geq 0$ , at least for all small  $|\lambda| > 0$ , the linearized equation (3.1) (and hence the KPP-4n one in (1.22) for  $f \approx 1$  for  $y \ll -1$ ) admits a 2D stable family of oscillatory solutions as  $y \rightarrow -\infty$ , and a 2D unstable one of exponentially divergent orbits.*

**3.2. Local blow-up and other instabilities to  $-\infty$ .** In order to verify the global continuation properties of stable bundles, one need to check whether the *nonlinear* ODE (1.22) admits blow-up and the dimension of such an unstable manifold. To this end, we re-write it down and, as usual, neglect the linear lower-order terms, which by standard local interior regularity are negligible for  $|f| \gg 1$ :

$$(3.4) \quad (|f|^n f)^{(4)} = -f^2 + f + \lambda f' = -f^2(1 + o(1)) \quad \text{as} \quad f \rightarrow \infty.$$

For any  $n \in [0, 1)$ , the unperturbed equation has the following exact blow-up solution: for the function  $F$  in (2.4),

$$(3.5) \quad F^{(4)} = -|F|^{\frac{2}{n+1}} \implies F_0(y) = -C_0(n)(Y_0 - y)^{-\frac{4(n+1)}{1-n}} \rightarrow -\infty \quad \text{as } y \rightarrow Y_0^-,$$

where  $Y_0 \in \mathbb{R}$  is a fixed arbitrary blow-up point and  $C_0(n) > 0$  is an easily computed constant. Studying the dimension of its stable manifold, as in [23, § 2.4], yields

**Proposition 3.2.** *For the ODE (1.22), with any  $n \in [0, 1)$ , the stable manifold of blow-up solutions is 1D depending on a single parameter being their blow-up point  $Y_0 \in \mathbb{R}$ .*

For  $n = 1$ , the asymptotic ODE (3.5) has the form

$$(3.6) \quad F^{(4)} = -|F| \implies F_0(y) = -e^{-y} \rightarrow -\infty \quad \text{as } y \rightarrow -\infty,$$

so, instead of having finite  $y$ -blow-up, one gets this negative exponential growth for  $y \ll -1$ , with a similar one dimension.

Finally, for  $n > 1$ , we have algebraically growing solutions:

$$(3.7) \quad F_0(y) = -C_1(-y)^{\frac{4(n+1)}{n-1}} \rightarrow -\infty \quad \text{as } y \rightarrow -\infty \quad (C_1 > 0).$$

#### 4. OSCILLATORY SOLUTIONS NEAR FINITE INTERFACES: 3D ASYMPTOTIC BUNDLE

This part of the asymptotic analysis is essentially different from that for the semilinear case  $n = 0$  in [23, § 2]; however, such results are already known [22], so we can omit some details.

Thus, for  $n > 0$ , we describe the generic oscillatory behaviour of solutions of (2.4) close to finite interfaces. According to pioneering results in [5, 7], since 1988, it was known that ODEs like (2.4) admit compactly supported solutions of changing sign near finite interfaces. Let  $f(y)$  vanish at the interface  $y \rightarrow y_0 > 0$ , so that  $f(y) \equiv 0$  for  $y > y_0$ . Then, making for convenience the reflection  $y \mapsto y_0 - y$ , with  $y > 0$  small enough, and keeping the leading first two terms in (2.4) for  $y \approx 0^+$ , after integration once we obtain an exponentially small perturbation of the following third-order equation (the sign “ $-$ ” on the right-hand side appears because of the reflection in  $y$ ):

$$(4.1) \quad F''' = -\lambda |F|^{-\frac{n}{n+1}} F, \quad y > 0, \quad F(0) = 0.$$

We next scale out the positive constant  $\lambda$  to get the ODE

$$(4.2) \quad F''' = -|F|^{-\frac{n}{n+1}} F, \quad y > 0.$$

We next describe oscillatory solution of changing sign of the ODE (4.2), with zeros concentrating at the given interface point  $y = 0^+$ . Let us mention again that oscillatory properties of solutions are a common feature of higher-order degenerate ODEs. We refer to first results in [5, 7, 22], to [14, 15] (thin film equations), and to [28, Ch. 3-5], where further examples can be found.

To this end, by the scaling invariance of (4.2), we look for its solutions of the form

$$(4.3) \quad F(y) = y^\mu \varphi(s), \quad s = \ln y, \quad \text{where } \mu = \frac{3(n+1)}{n} > 3 \text{ for } n > 0,$$

where  $\varphi(s)$  is the so-called *oscillatory component*. Substituting (4.3) into (4.2) yields the following third-order equation for  $\varphi(s)$ :

$$(4.4) \quad P_3(\varphi) = -|\varphi|^{-\frac{n}{n+1}}\varphi,$$

where  $P_k$  denote linear differential polynomials obtained by a simple recursion procedure (see [28, p. 140]), so that

$$\begin{aligned} P_1(\varphi) &= \varphi' + \mu\varphi, & P_2(\varphi) &= \varphi'' + (2\mu - 1)\varphi' + \mu(\mu - 1)\varphi, \\ P_3(\varphi) &= \varphi''' + 3(\mu - 1)\varphi'' + (3\mu^2 - 6\mu + 2)\varphi' + \mu(\mu - 1)(\mu - 2)\varphi. \end{aligned}$$

According to (4.3), we are interested in uniformly bounded global solutions  $\varphi(s)$  that are well defined as  $s = \ln y \rightarrow -\infty$ , i.e., as  $y \rightarrow 0^+$ . The best candidates for such global orbits of (4.4) are periodic solutions  $\varphi_*(s)$  that are defined for all  $s \in \mathbb{R}$ . Indeed, they can describe suitable (and, possibly, generic) connections with the interface at  $s = -\infty$ . See [22] for the following and other related results.

**Proposition 4.1.** *For any  $n > 0$ , (4.4) has a periodic solution  $\varphi_*(s)$  of changing sign.*

Two problems remain open:

- (i) uniqueness of the periodic solution  $\varphi_*(s)$ , and
- (ii) stability (hyperbolicity) of  $\varphi_*(s)$  as  $s \rightarrow +\infty$ .

Numerically, we have obtained positive answers to both questions. In particular, (i) and (ii) imply that there exists a unique (up to translation) periodic bounded connection with  $s = -\infty$ , where the interface is situated.

The convergence to the unique stable periodic solution of (4.4) is shown in Figure 6 for various  $n > 0$ . Different curves therein correspond to different Cauchy data  $\varphi(0)$ ,  $\varphi'(0)$ ,  $\varphi''(0)$  prescribed at  $s = 0$ . For  $n$  smaller than  $\frac{3}{4}$ , the oscillatory component gets extremely small, so an extra scaling is necessary, which is explained in [14, § 7.3]. A more accurate passage to the limit  $n \rightarrow 0$  in (4.4) is done there in Section 7.6 and in Appendix B. This explains the continuous deformation as  $n \rightarrow 0$  of oscillatory structures in (4.3) to linear ones in the exponential tail of the kernel  $F(y)$  of the fundamental solution of the bi-harmonic operator  $D_t + D_x^4$ .

In (d), we also present the periodic solution for  $n = +\infty$ , where (4.4) takes a simpler form (see an algebraic construction of the unique periodic solution in [14, § 7.4])

$$P_3(\varphi) = -\text{sign } \varphi.$$

Finally, given the periodic  $\varphi_*(s)$  of (4.4), as a natural way to approach the interface point  $y_0 = 0$  according to (4.3), we have that the ODE (4.2) generates at the singularity set  $\{f = 0\}$

(4.5) a 3D local asymptotic bundle with parameters  $y_0$ , a phase shift in  $s \mapsto s + s_0$ , and the parameter  $\varepsilon > 0$  of the scaling group for the ODE (4.2).

$$(4.6) \quad F \mapsto \varepsilon^{\frac{3(n+1)}{n}} F, \quad y \mapsto \varepsilon y \quad (\varepsilon > 0).$$

Notice that this scaling invariance has been lost in the approximate ODE (4.2).

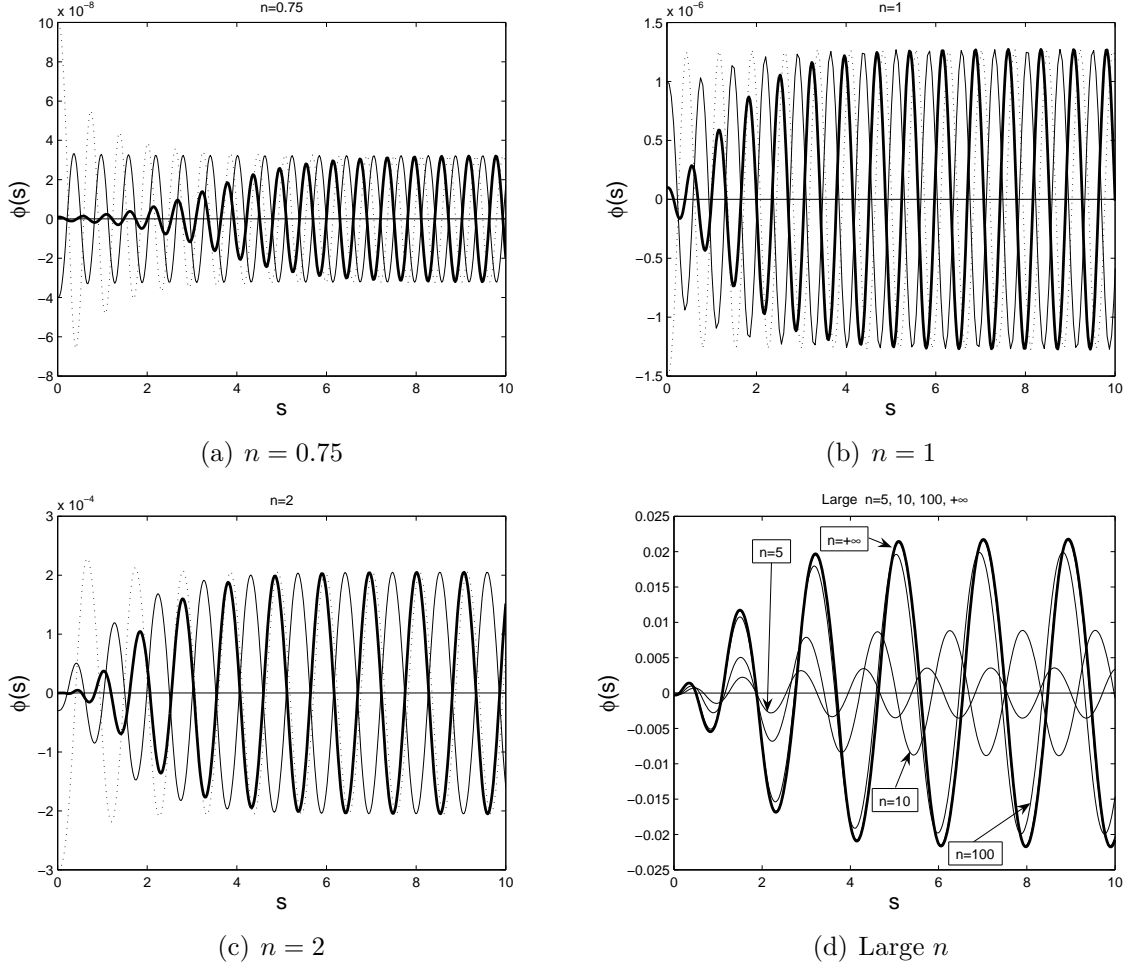


FIGURE 6. Convergence to a stable periodic orbit of the ODE (4.2) for various  $n > 0$ .

**Remark: nonnegative solutions for  $\lambda < 0$  (Example 3).** One can see that (4.4) does not admit nonnegative solutions and equilibria. However, if  $\lambda < 0$  in (4.1), then scaling out such  $\lambda$  leads to ODEs (4.2) and (4.4) with the opposite sign on the right-hand side, so that there exists a positive equilibrium

$$(4.7) \quad P_3(\varphi) = |\varphi|^{-\frac{n}{n+1}} \varphi \implies \varphi_0 = [\mu(\mu - 1)(\mu - 2)]^{-\frac{n+1}{n}}$$

(the rest of solutions are still oscillatory). Existence of such a particular positive solution in (4.7) is the “local” origin of the global nonnegative TW in Example 3 in Section 1, though such a behaviour is not structurally stable for both ODEs involved. Note also that the calculation (2.2) does not apply to (1.24) with a more complicated reaction  $q(f)$ .

**4.1. Well-defined matching of stable manifolds of equilibria 0 and 1.** Similar to [23] for  $n = 0$  (when the conclusion is indeed more straightforward), we thus observe that,

for  $n > 0$  and sufficiently small  $\lambda > 0$ , there is a well defined matching procedure of stable manifolds of equilibria  $f = 0$  (see (4.5)) and  $f = 1$  (Proposition 3.1). The overall relation of the dimensions is

$$(4.8) \quad 3_{(y \gg 1, f \rightarrow 0)} - 1 = 2_{(y \ll -1, f \rightarrow 1)},$$

where “ $-1$ ” stands for the dimension of the unstable (blow-up for  $n < 1$ ) manifold in Proposition 3.2. In other words, (4.8) implies a well posed algebraic system of two equations with two unknowns. In the case of analytic manifolds as in [23], this guarantees at most a countable number of solutions (or a finite one of uniformly bounded TW profiles).

In the present degenerate case  $n > 0$ , any analytic dependence on parameters is also plausible but difficult to prove. However, we think that (4.8) confirms that, for small  $\lambda > 0$ , the family of TW profiles is always discrete (what have been seen in numerical experiments), and then such (most probably, finite) number of  $\lambda$ -branches can be extended for larger  $\lambda > 0$  by classic nonlinear integral operator theory [39, 12]. Note that a fully global extension in  $\lambda$  is hardly possible: as was shown in [23] for  $n = 0$ , there is a clear phenomenon of existence of a maximal speed  $\lambda_{\max}$  such that, for  $\lambda > \lambda_{\max}$ , TW profiles  $f(y)$  are nonexistent. By a continuity argument, we expect that a similar phenomenon takes place for  $n > 0$ , but this is difficult to check even numerically.

## 5. A FEW WORDS ON “MORE QUASILINEAR” KPP- $4n$ PROBLEM

This is about the KPP-setting for the following quasilinear equation:

$$(5.1) \quad u_t = -(|u|^n u)_{xxxx} + |u|^n u(1 - |u|^n u) \implies -\lambda f' = -(|f|^n f)^{(4)} + |f|^n f(1 - |f|^n f),$$

where the source contains the same nonlinear term  $|u|^n u$  as the porous medium diffusion one. A one advantage of this model is that the corresponding ODE for  $F = |f|^n f$  is less complicated than that in (2.4):

$$(5.2) \quad F^{(4)} = \frac{\lambda}{n+1} |F|^{-\frac{n}{n+1}} F' + F(1 - F),$$

which, as might be expected, could improve convergence of numerical schemes. However, this did not happen, and convergence, though being slightly better, was not improved somehow essentially.

In Figures 7 and 8, we show TW profiles  $F(y)$  for  $n = 0, 0.5$ , and  $1$  for the two cases  $\lambda = 1$  and  $\lambda = 0.5$ .

An unusual application of such a quasilinear bi-harmonic equation (5.1) is that it allows to treat the “fast diffusion” case  $n < 0$ ; more precisely,  $n \in (-1, 0)$ . In Figure 9, we show such oscillatory profiles for  $n = 0, -0.25$ , and  $-0.5$  for  $\lambda = 0.5$ . The convergence for  $n < 0$  is much better than in the standard porous medium case  $n > 0$ . It is worth mentioning that, since for  $u \approx 0$  the source term becomes a non-Lipschitz function,  $Q(u) \sim +|u|^n u$ ,  $n < 0$ , there appears a standard non-uniqueness of solutions even in the simple ODE:

$$(5.3) \quad u_t = |u|^n u, \quad t > 0, \quad u(0) = 0 \implies \exists u(t) = (|n|t)^{\frac{1}{|n|}} > 0,$$

together with the trivial solution  $u(t) \equiv 0$ . In such cases, in PME and fast diffusion theory with blow-up, extinction, quenching, and other singularities, an extended semigroup of

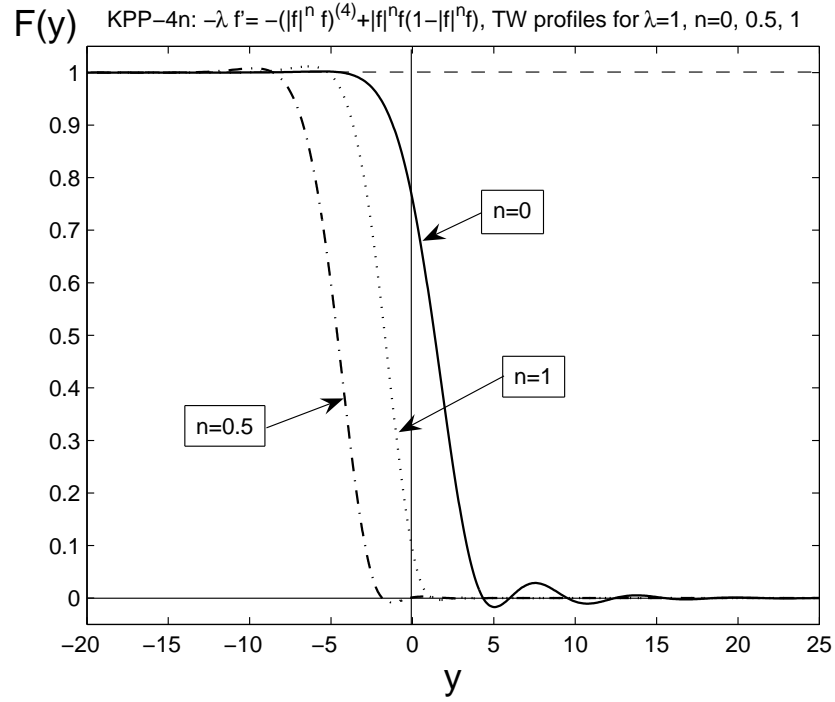


FIGURE 7. The TW profiles  $F(y)$  of (5.2), (1.23) for  $\lambda = 1$ ,  $n = 0, 0.5$  and  $1$ .

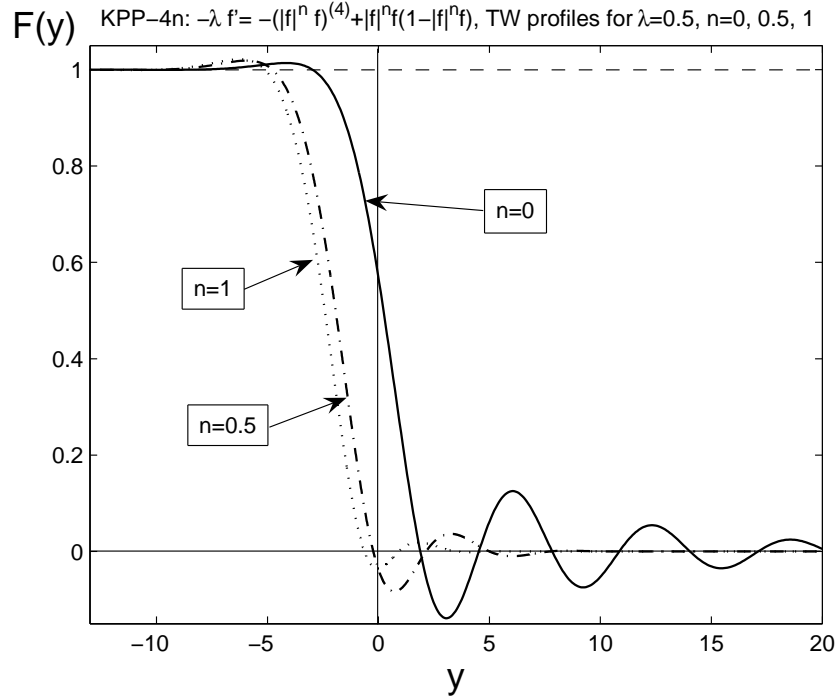


FIGURE 8. The TW profiles  $F(y)$  of (5.2), (1.23) for  $\lambda = 0.5$ ,  $n = 0, 0.5$ , and  $1$ .



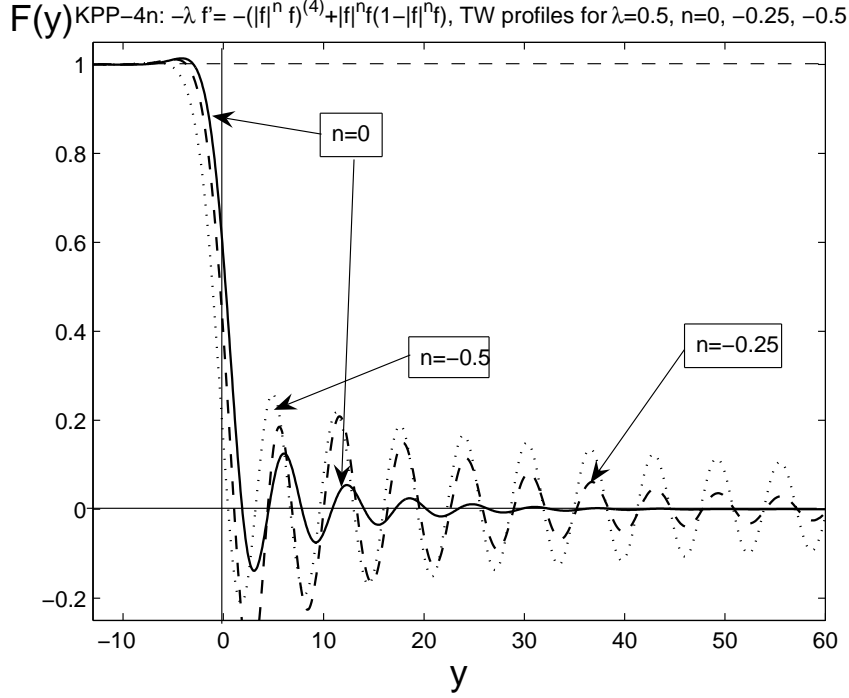


FIGURE 9. The TW profiles  $F(y)$  of (5.2), (1.23) in the “fast diffusion” case for  $\lambda = 0.5$ ,  $n = 0, -0.25$  and  $-0.5$ .

proper extremal (maximal or minimal) solutions is constructed; see [21, Ch. 6,7] and references therein. However, such a theory is fully nonexistent for higher-order parabolic flows with such singularities. Creating such a theory for equations with no Maximum Principle is a hard open problem.

## 6. A FEW WORDS ON NON-DIVERGENT KPP THIN FILM PROBLEM

We next briefly review some results for the KPP problem for the parabolic *thin film equation* (TFE-4) ( $n > 0$ )

$$(6.1) \quad u_t = -(|u|^n u_{xxx})_x + u(1 - u) \implies -\lambda f' = -(|f|^n f''')' + f(1 - f).$$

In Figure 10, we show TW profiles for  $\lambda = 1$  and sufficiently small  $n = 0.3, 0.6$ , and  $0.9$ , and, for comparison, we present the “linear” one for  $n = 0$  (i.e., for the KPP-4 problem in [23]). Next Figure 11 shows profiles for larger  $n = 1.5, 2, 3$ , and  $4$ .

In Figure 12, we show an enlarged image of the behaviour of  $f(y)$  from Figure 11 for large  $y \in [10, 17]$ . It is seen that  $f(y)$  becomes less oscillatory close to interfaces as  $n$  reaches about 2. We believe that a changing sign feature for  $n \geq 2$  in this figure can be related to the necessary regularization of the degenerate thin film operator in (6.1), where we replace

$$(6.2) \quad |f|^n \mapsto (\varepsilon^2 + f^2)^{\frac{n}{2}}, \quad \text{with } \varepsilon = 10^{-3},$$

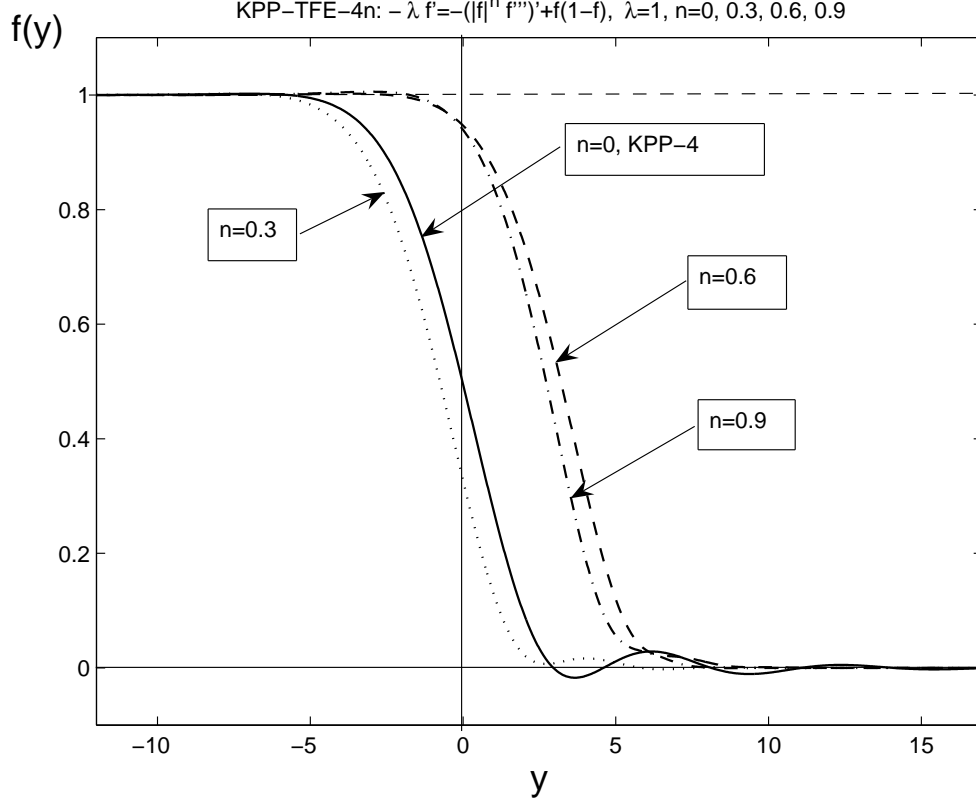


FIGURE 10. The TW profiles  $f(y)$  of (6.1), (1.23) for  $\lambda = 1$ ,  $n = 0, 0.3, 0.6$ , and  $0.9$ .

since smaller  $\varepsilon = 10^{-4}$ , etc., always led to divergence of the numerical scheme; see below.

Note that the convergence for larger  $n$  got very slow; for  $n = 4.3$ , we still got some profile, which is close to that for  $n = 4$  in Figure 11, while, for  $n \geq 4.4$ , “a singular Jacobian” (a full non-convergence) always appeared.

The oscillatory behaviour of  $f(y)$  near the interface was studied in [14, 25]. It was shown that a periodic oscillatory component  $\varphi(s)$  (a full analogy of that in (4.3), (4.4)) exists up to a critical *homoclinic bifurcation* exponent  $n_h$ ,

$$(6.3) \quad 0 < n < n_h \in \left(\frac{3}{2}, n_+\right), \quad \text{where} \quad n_+ = \frac{9}{3+\sqrt{3}} = 1.9019238\dots$$

Numerically,  $n_h$  is given by

$$(6.4) \quad n_h = 1.75987\dots$$

Therefore, it is expected that, for  $n > n_h$ ,  $f(y)$  exhibits either a finite number of zeros close to the interface, or even becomes nonnegative (at least, for  $n \geq 2$ ).

Finally, existence of a  $\log t$ -shift of moving front for the PDE KPP-TFE-4 problem (6.1) is shown precisely in the same way as in Section 9.

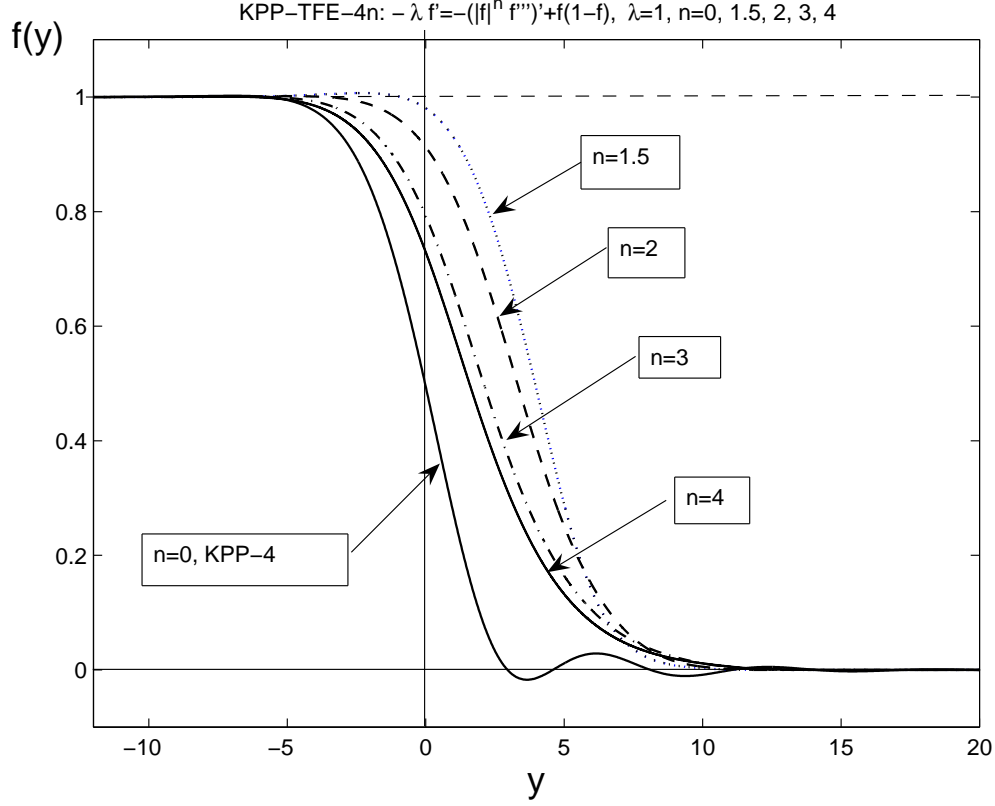


FIGURE 11. The TW profiles  $f(y)$  of (6.1), (1.23) for  $\lambda = 1$ ,  $n = 1.5, 2, 3$ , and 4.

## 7. THE QUASILINEAR KPP-(6,1) $n$ PROBLEM

We now, more briefly than above, consider the KPP-(6,1) $n$  problem (1.29) and its ODE counterpart

$$(7.1) \quad -\lambda f' = (|f|^n f)^{(6)} + f(1-f), \quad F = |f|^n f,$$

with singular boundary conditions (1.23).

Figure 13 shows TW profiles for  $\lambda = 1$  and  $n = 0.25$ . For comparison, we also put the “semilinear” profile for  $n = 0$ . This confirms a rather non-surprising fact that there exists a continuous dependence of  $f(y)$  on  $n \rightarrow 0^+$  (this even can be proved for such ODEs).

In Figure 14, we present a similar comparison for the more oscillatory case  $\lambda = 0.2$ , where we again observe existence of a clear “homotopy” deformation as  $n \rightarrow 0$ .

A proper dimensional analysis of the linearized bundle as  $y \rightarrow -\infty$  (i.e., as  $f \rightarrow 1$ ) is performed similarly, as in [23].

The oscillatory periodic-like behaviour at the finite interface as  $y \rightarrow y_0^-$  (i.e., as  $f \rightarrow 0$ ) is performed as in [15]; see also [28, p. 142].

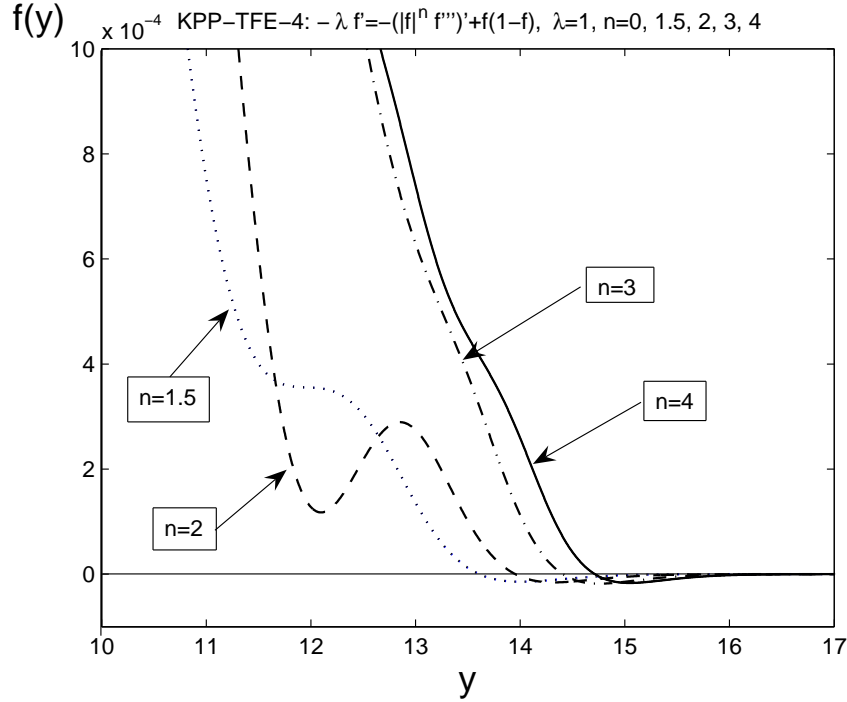


FIGURE 12. Enlarged behaviour of TW profiles  $f(y)$  close to zero of (6.1), (1.23) from Figure 11;  $\lambda = 1$ ,  $n = 1.5, 2, 3$ , and 4.

Note that the corresponding blow-up problem for  $n \in (0, 1)$  (cf. the ODE (3.5))

$$(7.2) \quad (|f|^n f)^{(6)} = f^2$$

is now more difficult since (7.2) admits oscillatory solutions, which can be studied as in [14, § 7] and in [15] by introducing an *oscillatory component* represented by periodic or other functions.

## 8. VERY BRIEFLY ON KPP-8n

In the KPP-8n, we deal with the following ODE problem:

$$(8.1) \quad u_t = -D_x^8(|u|^n u) + u(1 - u) \implies (\text{ODE}) \quad -\lambda f' = -(|f|^n f)^{(8)} + f(1 - f).$$

Figure 15 shows TW profiles  $F$  for  $\lambda = 0.5$  and  $n = 0.5, 1, 2$ , plus, for comparison, for  $n = 0$  from [23, § 4]. This shows a principal positive answer on the TW existence question. It is seen that oscillations about  $f = 0$  decrease as  $n$  increases, though finite interfaces for  $n > 0$  (unlike  $n = 0$  with an exponential decay as  $y \rightarrow +\infty$ ) are obviously invisible.

Stable bundles as  $f \rightarrow 1$  ( $y \rightarrow -\infty$ ) are studied as in [23], while, close to finite interfaces, as  $f \rightarrow 0$ , periodic oscillatory components for seventh-order ODEs are constructed as in [28, p. 142].

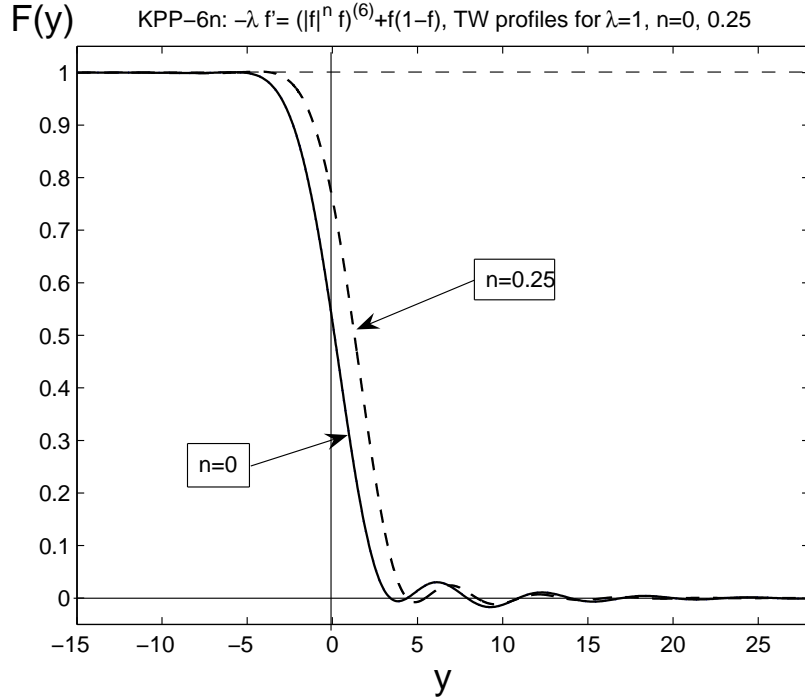


FIGURE 13. The TW profiles  $F(y)$  satisfying (2.4), (1.23) for  $\lambda = 1$ , and  $n = 0.25, n = 0$ .

## 9. THE ORIGIN OF $\log t$ -SHIFT: CENTRE SUBSPACE BALANCING

We now explain the origin of possible  $\log t$ -shiftings (say, retarding) from the TW for the PDE higher-order quasilinear parabolic KPP-problems (1.21) or (1.29). Note that the actual proof of such  $\log t$ -drift assumes a delicate matching of the solution behaviour on compact subsets in the TW  $y$ -variable, i.e., in the *Inner Region*, with a remote *Outer Region* for  $y \gg 1$ , where the influence of the nonlinear term  $-u^2$  is negligible, and the actual behaviour is governed by the quasilinear bi- (or tri-) harmonic operator. Such a matching, eventually, is supposed to describe classes of initial data, for which such  $(\pm)$   $\log t$ -shifting (or no shifting at all, say, a pure convergence to a TW in its moving frame). We do not present here this kind of a procedure of *matching* of asymptotic expansions of those two regions, which is extremely difficult in the quasilinear/degenerate case. Therefore, we restrict to an Inner expansion analysis.

**9.1. Linearization and rescaled equation.** Thus, we consider a KPP-type problem for a quasilinear PDE

$$(9.1) \quad u_t = \mathbf{A}(u) + u(1 - u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

with some proper step-like data  $u_0(x)$ , with, however, some positive, negative, or even oscillatory “tails” as  $y \rightarrow +\infty$ , in order to create a necessary  $\pm \log t$ -shift. Here, in (9.1),  $\mathbf{A}(u)$  is a proper homogeneous isotropic quasilinear differential operator satisfying

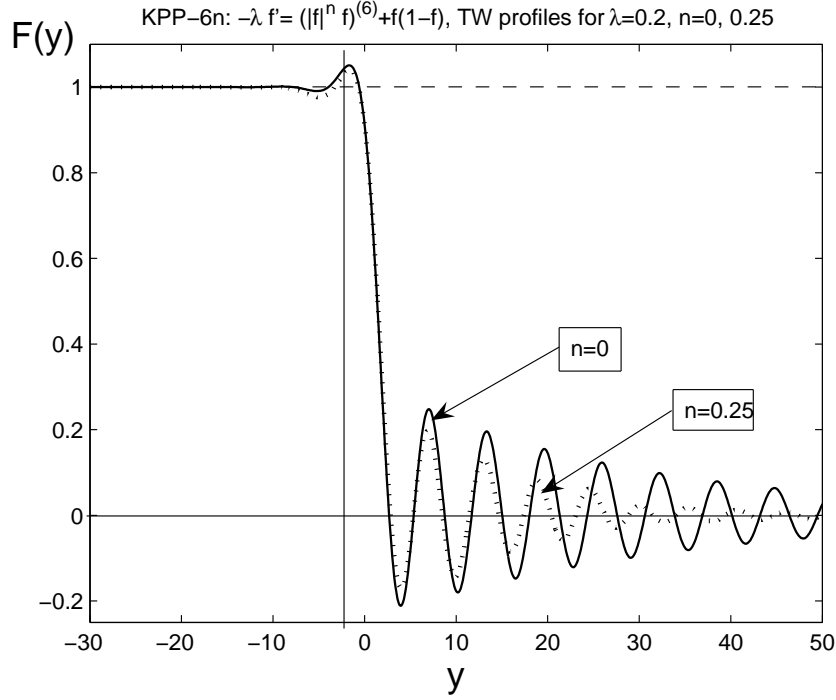


FIGURE 14. The TW profiles  $F(y)$  satisfying (2.4), (1.23) for  $\lambda = 0.2$ , and  $n = 0.25$ ,  $n = 0$ .

some extra conditions specified below. We fix, as key examples, the quasilinear bi- or tri-Laplacian operators of the *porous medium type*

$$(9.2) \quad \mathbf{A}(u) = -D_x^4(|u|^n u), \quad \text{or} \quad \mathbf{A}(u) = D_x^6(|u|^n u), \quad n > 0.$$

We assume that, for some fixed  $\lambda_0 > 0$ , the corresponding ODE problem

$$(9.3) \quad -\lambda_0 f' = \mathbf{A}(f) + f(1 - f),$$

with the conditions (1.23) admits a unique solution  $f$ .

Attaching the solution  $u(x, t)$  to the front moving and setting, for convenience,  $x_f(t) \equiv \lambda_0 t - g(t)$ , the PDE reads

$$(9.4) \quad u(x, t) = v(y, t), \quad y = x - \lambda_0 t + g(t) \implies v_t = \mathbf{A}(v) + v(1 - v) + \lambda_0 v_y - g'(t)v_y.$$

We next linearize (9.4) by setting

$$(9.5) \quad v(y, t) = f(y) + w(y, t),$$

that yields the following perturbed equation:

$$(9.6) \quad w_t = \mathbf{B}w - g'(t)f' - g'(t)w_y - w^2, \quad \text{where} \\ \mathbf{B}w = \mathbf{A}'(f)w + (1 - 2f)w + \lambda_0 w_y \quad \text{and} \quad \mathbf{A}'(f)w = (n + 1)\mathbf{A}(|f|^n w).$$

Assuming that, in this  $g(t)$ -moving frame, there exists the convergence as in (1.12), so that  $w(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , one can see that the leading non-autonomous perturbation in

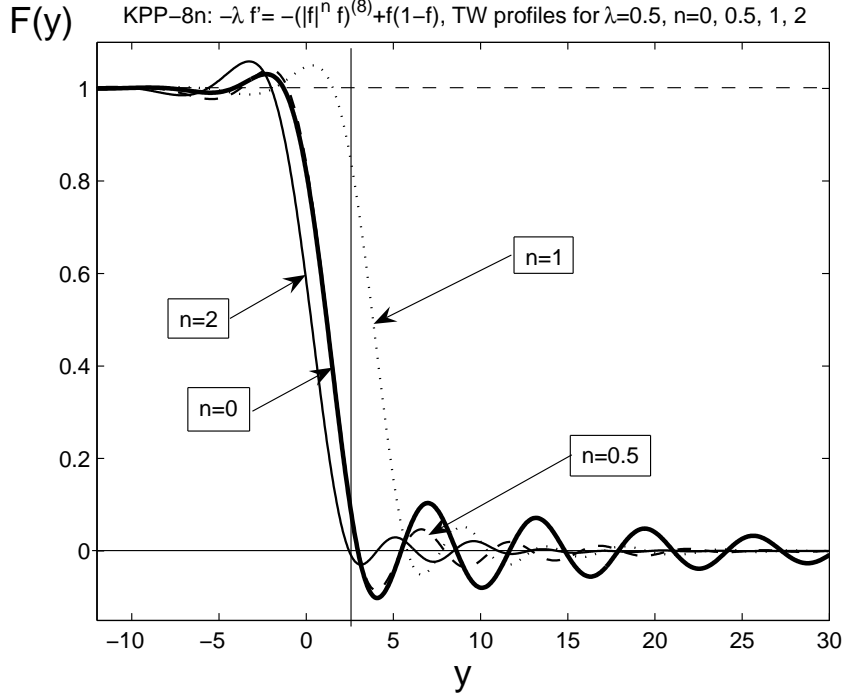


FIGURE 15. The TW profiles  $f(y)$  satisfying (8.1), (1.23) for  $\lambda = 0.5$  and  $n = 0, 0.5, 1, 2$ .

(9.6) is the second term on the right-hand side. However, as we show, the last two terms, though negligible, will define a proper  $\log t$ -shift of the front.

**9.2. Where  $\log t$ -shift comes from.** Again, as in the semilinear case  $n = 0$ , we note that the rescaled equation (9.6) is essentially *non-autonomous* in time, so we cannot use powerful tools of nonlinear semigroup theory; see [40]. However, using a formal asymptotic approach, we will trace out some definite *centre subspace* behaviour after an extra rescaling and balancing of non-autonomous perturbations.

Thus, as usual (see Introduction), we assume that  $g'(t) \rightarrow 0$  as  $t \rightarrow +\infty$  sufficiently fast, i.e., at least algebraically, so that

$$(9.7) \quad |g''(t)| \ll |g'(t)| \quad \text{for } t \gg 1.$$

Under the hypothesis (9.7), the only possible way to balance *all* the terms therein (including the quadratic one  $-w^2$ ) for  $t \gg 1$  is to assume the asymptotic separation of variables:

$$(9.8) \quad w(y, t) = g'(t)\psi(y) + \varepsilon(t)\varphi(y) + \dots, \quad \text{where } |\varepsilon(t)| \ll |g'(t)| \quad \text{as } t \rightarrow +\infty.$$

Here, we omit higher-order perturbations. Substituting (9.8) into (9.8) yields

$$(9.9) \quad \begin{aligned} g''(t)\psi + \varepsilon'(t)\varphi + \dots &= g'(t)(\mathbf{B}\psi - f') \\ &+ \varepsilon(t)\mathbf{B}\varphi - (g'(t))^2(\psi' + \psi^2) - g'(t)\varepsilon(t)(\varphi' + 2\varphi\psi) - \varepsilon^2(t)\varphi^2 + \dots \end{aligned}$$

Using (9.7) and (9.8) in balancing first the leading terms of the order  $O(g'(t))$  yields the elliptic equation for  $\psi$ :

$$(9.10) \quad O(g'(t)) \left( = O\left(\frac{1}{t}\right) \right) : \quad \mathbf{B}\psi - f' = 0.$$

Then balancing the rest of the terms in (9.9) requires their asymptotic equivalence,

$$(9.11) \quad g''(t) \sim -(g'(t))^2 \sim \varepsilon(t), \text{ i.e., } g(t) = k \log t, \quad g'(t) = \frac{k}{t}, \quad g''(t) = -\frac{k}{t^2}, \quad \varepsilon(t) = \frac{1}{t^2}.$$

Then, we obtain the second inhomogeneous singular Sturm–Liouville problem for  $\varphi$ :

$$(9.12) \quad O\left(\frac{1}{t^2}\right) : \quad \mathbf{B}\varphi = k\psi + k^2(\psi' + \psi^2).$$

Thus, the first simple asymptotic ODE in (9.11) gives the  $\log t$ -dependence as in (1.13). Finally, we arrive at the following system for  $\{\psi, \varphi\}$ :

$$(9.13) \quad \begin{cases} \mathbf{B}\psi = f', \\ \mathbf{B}\varphi = k\psi + k^2(\psi' + \psi^2). \end{cases}$$

Solving this system, with typical boundary conditions as in (1.23), allows then continue the expansion of the solutions of (9.6) close to an “affine (i.e., shifted via  $f'$  on the RHS) centre subspace” of  $\mathbf{B}$  governed by the spectral pair obtained by translation in (9.3):

$$(9.14) \quad \hat{\lambda}_0 = 0 \quad \text{and} \quad \hat{\psi}_0(y) = f'(y).$$

The asymptotic expansion for  $t \gg 1$  then takes the form

$$(9.15) \quad w(y, t) = \frac{k}{t} \psi(y) + \frac{1}{t^2} \varphi(y) + \dots,$$

which can be easily extended by introducing further terms, with similar inhomogeneous Sturm–Liouville problems for the expansion coefficients.

As for  $n = 0$ ,  $\mathbf{B}$  does not have a discrete spectrum, so we cannot get a simple algebraic equation for  $k$  by demanding the standard orthogonality of the right-hand side in the second equation in (9.13) to the adjoint eigenvector  $\hat{\psi}_0^*$  of  $\mathbf{B}^*$  in some “weighted” metric  $\langle \cdot, \cdot \rangle_*$  (in which the adjoint operator  $\mathbf{B}^*$  is obtained, if any), like

$$(9.16) \quad k : \quad \langle k\psi + k^2(\psi' + \psi^2), \hat{\psi}_0^* \rangle_* = 0.$$

Therefore, it seems, the system (9.13) cannot itself determine the actual value of  $k$  therein. As we have mentioned, the latter requires a difficult matching analysis in Inner and Outer Regions, which, for the KPP–4n (and all other problems) remains an open problem.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, BATH BA2 7AY, UK  
*E-mail address:* masvg@bath.ac.uk